

Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble

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Outline

- Introduction
 - Random matrices, the Wishart ensemble
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Wishart-Laguerre Ensemble

- We consider Wishart ensemble (Biometrika, 1928)
- Distribution of the $M \times N$ matrix X is Gaussian

$$P(X) \propto \exp \left[-\frac{1}{2} \text{Tr} X \right]$$

Density of Eigenvalues

- From distribution of Wishart matrices joint distribution of N eigenvalues

$$N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_0} e^{-\frac{1}{2} \sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{Y/2} \prod_{j < k} |\lambda_j - \lambda_k|^\beta$$

- First interesting object of study is the spectral density

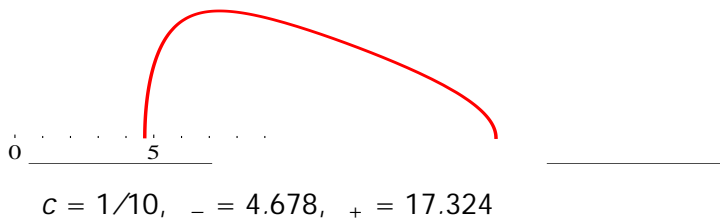
$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i)$$

The Marčenko-Pastur Law

- For large N , $\rho_N(x) = (1/N)f(x/N)$ follows the Marčenko-Pastur law (1967)

$$f(x) = \frac{\rho \sqrt{(x_+ - x)(x - x_-)}}{2x} \Big|_{x \in [x_-, x_+]}, \quad \begin{aligned} - &= \frac{1}{3\sqrt{c}} - 1 \sqrt{\frac{2}{c}}, \text{ (hard edge)} \\ + &= \frac{1}{\sqrt{c}} + 1 \sqrt{\frac{2}{c}}, \text{ (soft edge)} \end{aligned}$$

$$c = \frac{N}{M}$$



The Smallest Eigenvalue

Mathematics

- invertibility of Wishart matrix is controlled by λ_{\min}
- Compressive sensing: fluctuations of λ_{\min} set bounds on # of measurements to fully recover a sparse signal

Statistics

- Statistical tests based on W^{-1} (e.g. Hotelling's T -square test)

Physics

- Quantum information -measure of entanglement

The Smallest Eigenvalue

Exact expressions for finite N and M using various techniques, e.g.

- Edelman's approach (1991)

$$\lambda_{\min}^{(M,N)} = C_{M,N}^{(M-N-1)/2} e^{-\lambda N/2} g_{M,N}(\lambda)$$

with $g_{M,N}(\lambda)$ polynomials (different expressions for $M - N$ even or odd).

These expressions (and similar ones) difficult to evaluate for large sizes.

For large N , information on the typical fluctuations of the smallest eigenvalue ($c < 1$): Tracy-Widom distribution (Feldheim & Sodin, 2010)

$$\lambda_{\min} = -c^{1/6} N^{-2/3} \beta, \quad \beta \sim TW_{\beta}$$

Our Goal

Study large fluctuations of the smallest eigenvalue

- simple expressions for rate functions for large deviations.

$$P_N^{(\min)}(t) \sim e^{-\beta N^2 \phi_+^{(\min)}\left(\frac{t-N_-}{N}\right)}, \quad N_- \leq t <$$

$$P_N^{(\min)}(t) \sim e^{-\beta N \phi_-^{(\min)}\left(\frac{N_- - t}{N}\right)}$$

Coulomb Gas approach

- From joint distribution of eigenvalues

$$N(\lambda) = \frac{1}{Z_0} e^{-\frac{\beta}{2} \sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{M-N-1} \prod_{j < k} |\lambda_j - \lambda_k|^\beta$$

- Coulomb Gas: eigenvalues as a system of charged particles in a 2D world (logarithmic potential), constrained to the real line and external linear-log potential

$$N(\lambda) = \frac{e^{-\beta F(\cdot)/2}}{Z_0}$$

with

$$F(\lambda) = \sum_{i=1}^N \lambda_i^2 - \mu \sum_{i=1}^N \lambda_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \log |\lambda_i - \lambda_j|$$

Coulomb Gas approach

- Quantity to calculate:

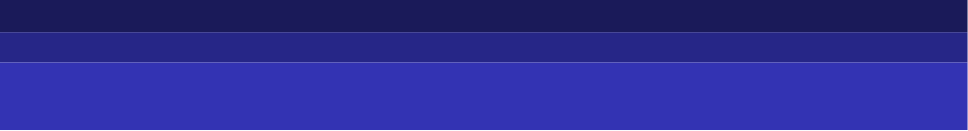
$$P_N^{(\min)}(t) = \text{Prob}(\min_{1 \leq i \leq N} \lambda_i \geq t) = \frac{Z(t)}{Z_0}$$

with

$$Z(t) = \int_t^\infty \dots \int_t^\infty e^{-\frac{1}{2}F(\lambda)} d\lambda_1 \dots d\lambda_N$$

and $Z_0 = Z(t = 0)$.

- Coulomb gas with hard wall at t .



Analytics - Path integral

To obtain

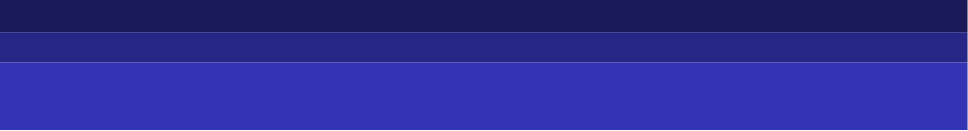
$$Z(t) = \int Df e^{-\frac{1}{2} N^2 S[f(x)]}$$

with

$$S[f(x)] = \int dx f(x) x^{-\mu} + \frac{-2}{N} \int dx f(x) \log x - \int dx dy f(x) f(y) \log |x - y| + \frac{2}{N} \int dx f(x) \log f(x) + C_1 \int dx f(x) - 1$$

with $\mu = (1 - c)/c$.

- No Dyson correction in the entropic term



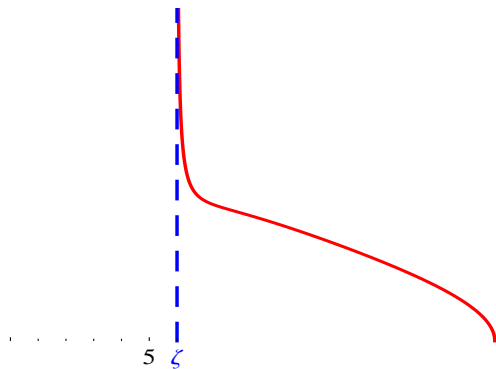
Analytics: Finite Interval Hilbert Transformation

Solution (Mathematical solution + normalisation + positivity):

An intuitive representation

An intuitive representation

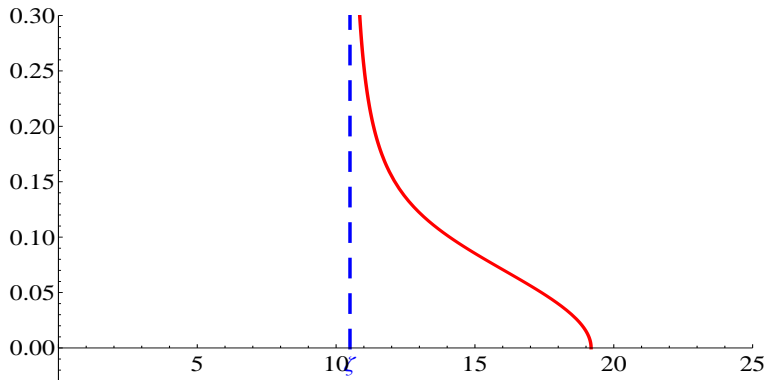
An intuitive representation



An intuitive representation



An intuitive representation



Large deviations to the left of λ_{\min}

- Coulomb Gas approach (as presented) not able to capture fluctuations to the left of λ_{\min}
- Reason: we only consider leading terms $O(N^2)$, which capture bulk properties

Large deviations to the left of λ_{\min}

- Energetic Argument (Majumdar & Vergassola)
- Expression the free energy $F(\lambda)$
- Energetic cost of moving the smallest eigenvalue to the left $t = -N$ (this does not require a global rearrangement of the bulk)

$$\begin{aligned}
 E(t) &= F(t, \lambda_2, \dots, \lambda_N) - F(-N, \lambda_2, \dots, \lambda_N) \\
 &= t - N \log(t) - 2 \sum_{k=2}^N \log |t - \lambda_k| + C \\
 &= t - N \log(t) - 2N \int_{\text{MP}(\lambda)} d\lambda \log |t - \lambda| + C
 \end{aligned}$$

C so that $E(t = -N) = 0$.

Large deviations to the left of λ_{\min}

- Obtain

$$P_N^{(\min)}(t) = e^{-\beta N} \left(\frac{N-t}{N} \right)^{N-t}, \quad 0 \leq t \leq N$$

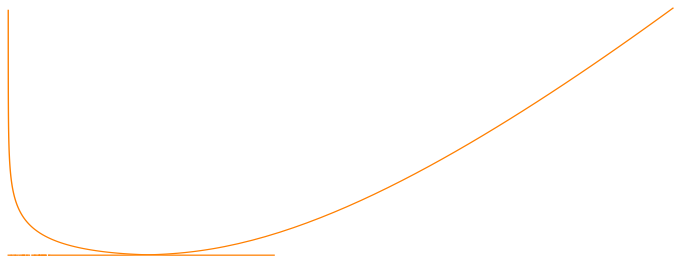
- Left rate function

$$\begin{aligned} \mu_{-}^{(\min)}(x) = & -\frac{1}{2} \log \left(1 - \frac{x}{\bar{x}} \right) - \frac{1}{2} \log \frac{1}{x(x + \bar{x})} \\ & + 2 \log \frac{\bar{x}}{x + \bar{x}} \\ & + \log \left(1 + 2 \frac{\bar{x}}{x(x + \bar{x}) - x} \right), \quad 0 \leq x \leq \bar{x} \end{aligned}$$

with $\bar{x} = \frac{1}{4} \left(1 + \sqrt{5} \right)$.

Large deviations- Numerics

$N = 11, M = 110$. Comparison with Edelman's (91) for $\beta = 1$



Comparison with Tracy-Widom

$$P_N^{(\min)}(t) \sim \lim_N P_{\beta, N} \left(\min - z_N^{(\beta)} \right) / S_N^{(\beta)} \quad t$$

To compare with Tracy-Widom, expand rate functions:

$$P_N^{(\min)}(x) \sim \frac{2}{3} c^{1/4} x^{3/2}, \quad P_N^{(\min)}(x) \sim \frac{1}{24} \frac{1}{c} x^3$$

Then

$$P_N^{(\min)}(t) \sim \begin{cases} \exp \left(-\frac{2}{3} \zeta_{-}^{3/2}(t) \right), & 0 \leq t \leq -N \\ \exp \left(-\frac{1}{24} \zeta_{-}^3(t) \right), & t > -N \end{cases}$$

with $\zeta_{-}(t) = -\frac{N\zeta_{-} - t}{N^{1/3} \zeta_{-}^{2/3} c^{1/6}}$

Almost Square Matrices

- $M = N + a$, $\beta = a/N$, $a \rightarrow \infty$, $a(\beta) = a + (N - 2)/\beta$

- Look at the behaviour for $z = Nt$

$$P_N^{(\min)}(z) \begin{cases} < \exp\left(-a - \frac{4z}{a^2}\right), & z \in [0, a^2/4] \\ \sim \exp\left(-a^2 + \frac{4z}{a^2}\right), & z \in [a^2/4, \infty) \end{cases}$$

with

$$P_N^{(\min)+}(x) = \frac{1}{8} \left(x - 4 \sqrt{x} + 3 + \ln x \right),$$

$$P_N^{(\min)-}(x) = \ln \frac{1 + \sqrt{1-x}}{x} - \sqrt{1-x}$$

- $a(\beta) = 0$ ($a = 1$, $\beta = 1$ or $a = 0$, $\beta = 2$)

$$P_N^{(\min)}(z) = e^{-\beta z/2}$$

Almost Square Matrices

Comparison with Edelman's exact result for $\rho = 1$ ($N = 200$,
 $a=5$)

Subleading contributions

- Entropic contribution: Saddle-point equation

$$\frac{1}{2} (x - \log x) + \frac{1}{N} \log f(x) + D = \int_{\zeta}^Z dy f(y) \log |x - y|$$

Support of $f(x)$ is not compact fluctuations to the left of
min

- Non-linear integral equation (Hammerstein type)
- Standard perturbation is hopeless
- Non-standard perturbation (boundary layer theory ?) as difficult as the original equation

Subleading contributions

Two options:

- simplest analytical approach: $y \in R_{\text{interior}}, x \in R_{\text{exterior}},$
 $V(x) = \frac{1}{2} (x -)^2$

Subleading contributions

■ Numerical solution (Abdou & Ismail 2002)

