Rank 1 real Wishart spiked model

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- **Real Wishart spiked models**
- **Contour integral formula for j.p.d.f.**
- **Asymptotic analysis and phase transition**

Real Wishart spiked models

- Real Wishart matrix S in $W_{\mathbb{R}}(\Sigma, M)$:

1. X: $N \times M$ $(M \geq N)$ and columns of X are i.i.d. N-variate normal variables with zero mean.

2. Σ: covariance matrix $\Sigma_{ij} = E(X_{i1}X_{j1})$.

- Wishart matrix:

$$S = \frac{1}{M} X X^T,$$

- Think of each column of X as a sample from a N-variate normal variables with zero mean. Then S is the sample covariance matrix.

Previous results

- For complex and quartonionic Wishart matrices, the phase transition in rank 1 spiked model was studied by Baik, Ben-Arous and Péché (complex) and Wang (quartonionic).

- Let $1+\tau$ be the non-trivial eigenvalue and $\gamma = M/N$. Then for $-1 < \tau < \gamma^{-1}$, the largest eigenvalue distribution are same as the ones with $\Sigma = I$. i.e. Tracy-Widom distribution for the respective symmetry.

- Phase transition occurs at $\tau = \gamma^{-1}$. Largest eigenvalue distribution: Tracy-Widom GOE for both the complex and quartonionic case.

$$P\left(\left(\lambda_{max} - \left(\frac{\gamma+1}{\gamma}\right)^2\right)\frac{\gamma(2M)^{\frac{2}{3}}}{(1+\gamma)^{\frac{4}{3}}} < \zeta\right) = GOE(\zeta)$$

- Real case is more complicated. Very recently, Bloemendal and Virág characterized the distribution at the phase transition by a boundary value problem. They use stochastic operator method that is very different from ours.

- Our work uses orthogonal polynomi[illegible]thod to find th[illegible] distribution at the phase transitio[illegible] The [illegible]ult takes [illegible]e following form

$$P\left(\left(\lambda_{max} - \left(\frac{\gamma+1}{\gamma}\right)^2\right)\frac{\gamma(2[illegible])^{\frac{2}{3}}}{(1+[illegible])^{\frac{4}{3}}} < \zeta\right) = [illegible]\zeta),$$

$F(\zeta)$ is of the form

$$F(\zeta) = \sqrt{\quad} GUE(m/Z \; [illegible] \; x$$

- The functions φ_0, σ, φ_1 and ψ are all known, φ_0 σ and φ_1 are expressible in terms of the Hastings-McLeod solution to Painlevé II, and ψ is

$$\psi = (T - \tilde{\tau}\zeta)^{\frac{1}{2}}, \quad \tilde{\tau} = \frac{\tau}{2(1+\tau)},$$

$$\varphi_0''(\zeta) = \zeta\varphi_0(\zeta) + 2\varphi_0^3(\zeta),$$

$$\varphi_0(\zeta) \sim -\mathrm{Ai}(\zeta), \quad \zeta \to +\infty,$$

$$\sigma(\zeta) = \int_\zeta^\infty \varphi_0^2(u)\,du$$

Contour integral formula for j.p.d.f.

- Difficulty in finding the j.p.d.f.

$$P(\lambda) = \frac{1}{Z_{M,N}} |\Delta(\lambda)| \prod_{j=1}^{N} \lambda_j^{\frac{M-N-1}{2}} \int_{O(N)} e^{-\frac{M}{2}\mathrm{Tr}(\Sigma^{-1} g S g^{-1})} g^T dg,$$

If Σ has only one non-trivial eigenvalue $1+\tau$, then the integral is given by

$$\int_{\Gamma} e^{Mt} \prod_{j=1}^{N} e^{-\frac{M}{2}\lambda_j} \left(t - \frac{\tau}{2(1+\tau)} \lambda_j \right)^{-\frac{1}{2}} dt$$

This is similar to a formula by Bergére and Eynard

$$\int_{O(N)} e^{-\mathrm{Tr}\left(X g Y g^{-1}\right)} g^T dg \propto \int \frac{e^{\mathrm{Tr}(S)}}{\prod_{j=1}^{N} \det(S - y_j X)} dS$$

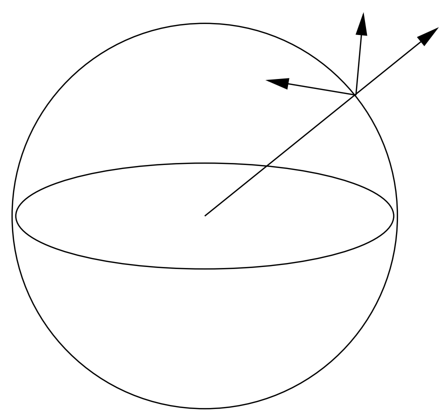
over i times real symmetric matrices.

- First, a simple computation shows

$$\int_{O(N)} e^{-\mathrm{Tr}\left(XgYg^{-1}\right)} g^T dg \propto \prod_{j=1}^{N} e^{-\frac{M}{2}\lambda_j} \int_{SO(N)} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^{N} \lambda_j g_{jN}^2} g^T dg$$

where g_{jN} is the last column of g.

- We can think of $SO(N)$ as the space of orthonormal frames.



- Can identify the last column of g as a point in S^{N} and the rest of the columns as a point in $SO(N \quad 1)$. E.g. if $Gg\hat{N} = (0, 0, \ldots, 1)$, then Gg is of the form

$$Gg = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$$

$V \in SO(N \quad 1)$. $V \quad (N$

- We can use this to compute the integral

$$\int_{O(N)} e^{-\mathrm{Tr}\left(XgYg^{-1}\right)} g^T dg \propto \prod_{j=1}^{N} e^{-}$$

then

$$\int_0^\infty e^{-st} I(\Sigma, \Lambda, t)dt \propto \prod_{j=1}^{N} e^{-\frac{M}{2}\lambda_j} \int_{\mathbb{R}^N} e^{\sum_{j=1}^{N}\left(-s+\frac{\tau M}{2(1+\tau)}\lambda_j\right) x_j^2} dX$$

Can be evaluated as

$$\int_0^\infty e^{-st} I(\Sigma, \Lambda, t)dt \propto \prod_{j=1}^{N} e^{-\frac{M}{2}\lambda_j} \left(s - \frac{\tau M}{2(1+\tau)}\lambda_j\right)^{-\frac{1}{2}}$$

- Taking the inverse Laplace transform, we obtain the contour integral formula

$$I(\Sigma, \Lambda) \propto \int_\Gamma e^{Mt} \prod_{j=1}^{N} e^{-\frac{M}{2}\lambda_j} \left(t - \frac{\tau}{2(1+\tau)}\lambda_j\right)^{-\frac{1}{2}} dt,$$

- To derive the formula, we first decompose the Haar measure into two parts, integrate out the part we do not need and then use the Laplace transform to ‘flatten’ to measure on S^{N-1}.

- The idea is similar to Bergére and Eynard, in which the whole Haar measure is 'flatten' by an integral transform to obtain an integral formula over the space of $N \times N$ symmetric matrices.

$$\int_{O(N)} e^{-\mathrm{Tr}\left(XgYg^{-1}\right)} g^T dg \propto \int \frac{e^{\mathrm{Tr}(S)}}{\prod_{j=1}^{N} \det(S - y_j X)} dS$$

Asymptotic analysis

- The contour integral expression reduces the problem to the analysis of the orthogonal ensemble with weight w

$$w(x) = e$$

$\mathcal{U}$ is the moment matrix with entries

$$\frac{1}{2}\int_0^\infty\int_0^\infty r_j(x)\,\mathrm{sgn}(x-y)\,r_k(y)\,w(x)\,w(y)\,dx\,dy,$$

y) dxdy;

1.9 Td[(i)2 0.5(2(5)1.1(6)1.1(.)1.4486.2]TJ(R)0.0.5(1)(9)0.0.7(0)5(T)R.1.4.5

.7852(i)-0.55(r)1.4572214(g)1.7844=(n)-67712Tf 9.48444452(i)-.0594

- The kernel S_1 can be written in terms of Laguerre polynomials.

$$S_1 = K_1 + K_2,$$

$$K_2 = \left(\frac{y(t-\tilde{t}y)}{x(t-\tilde{t}x)}\right)^{\frac{1}{2}} w_0^{\frac{1}{2}}(x) w_0^{\frac{1}{2}}(y) \frac{L_N(x)L_{N-1}(y) - L_N(y)L_{N-1}(x)}{h_{N-1,0}(x-y)},$$

$$K_1 = \left(\pi_{N+1,1}w \quad \pi_{N,1}w\right)(y) \begin{pmatrix} 0 & -\frac{M\tilde{t}}{2h_{N-1,0}} \\ -\frac{M\tilde{t}}{2h_{N-2,0}} & \frac{Mt-\tilde{t}(M+N)}{2h_{N-1,0}} \end{pmatrix} \begin{pmatrix} L_{N-2}(x) \\ L_{N-1}(x) \end{pmatrix} w$$

L_N: monic Laguerre polynomials orthogonal to the weight $x^{M-N}e^{-Mx}$.

- Asymptotics of S_1 can be computed using the asymptotics of the Laguerre polynomials. The moment matrix is also related to S_1.

$$\frac{\partial}{\partial t}\log\det\mathcal{U} = -\int_{\mathbb{R}_+}\frac{S_1(x,x)}{t-\tilde{\tau}x}dx,$$

- This gives us the following representation for the largest eigenvalue distribution.

$$\mathbb{P}(\lambda_{max} < z) \propto \int_{\Gamma}\exp\left(Mt - \frac{1}{2}\int_{c_0}^{t}\int_{\mathbb{R}_+}\frac{S_1(x,x)}{s-\tilde{\tau}x}dxds\right)\sqrt{\det\left(I - K\chi_{[z,\infty)}\right)}dt$$

Phase transition

- In the large N limit, both the Fredholm determinant and the integral

$$\int_{c_0}^{t}\int_{\mathbb{R}_+}\frac{K_1(x,x)}{s-\tilde{t}x}dxds$$

remains finite. The t integral in the largest eigenvalue distribution can therefore be computed using saddle point analysis for

$$Mt-\int_{c_0}^{t}\int_{\mathbb{R}_+}\frac{K_2(x,x)}{s-\tilde{t}x}dxds$$

Note that $K_2(x,x)$ is the same as the CD kernel for the Laguerre polynomials.

$$K_2(x,x) = w^{\frac{1}{2}}$$

and we have

$$K_2(x,x) \sim N\rho, \quad \int_{c_0}^{t} \int_{\mathbb{R}_+} \frac{K_2(x,x)}{s-\tilde{\tau}x} dx ds \sim N \int_{\mathbb{R}_+} \rho \log(t-\tilde{\tau}x) dx.$$

- Saddle point analysis gives (e.g. for $\gamma = 1$)

$$\left(1-\frac{1}{4\tilde{\tau}}\right)\left(1-\sqrt{\frac{t-4\tilde{\tau}}{t}}\right) = 0.$$

Then the saddle point is at

$$\frac{1}{2-4\tilde{\tau}}, \qquad \tilde{\tau} = \frac{\tau}{2(1+\tau)}, \qquad -\infty < \tilde{\tau} < \frac{1}{2}$$

and

$$\frac{t}{\tilde{\tau}} = \frac{1}{(2-4\tilde{\tau})\tilde{\tau}}.$$

- The function $t/\tilde{\tau}$ decreases from 0 to $-\infty$ for $\tilde{\tau} < 0$ and is greater than or equal to 4 for $\tilde{\tau} > 0$.

- When $\tilde{\tau} = 1/4$, $\tau = 1$ we have $t/\tilde{\tau} = 4$ and the saddle point coincide with the edge point of spectrum. This gives a phase transition.

- When $\tilde{\tau} \to \infty$, $t/\tilde{\tau} \to 0$, $\Rightarrow$ different behavior for smallest eigenvalue.

- **When $\tau = 1$, the singularity $t/\tilde{\tau}$**

$$K_2 \to \left(\frac{T - \tilde{\tau}\xi_2}{T - \tilde{\tau}\xi_1}\right)^{\frac{1}{2}} \frac{Ai(\xi_1)Ai'(\xi_2) - Ai'(\xi_1)Ai(\xi_2)}{\xi_1 - \xi_2}$$

- Contribution of each term to asymptotics:

$$\det(I - \chi K \chi) = \det((I - K_2\chi)^{-1}) \det(I - K_c)$$

first term gives Tracy-Widom GUE independent on T. Second term is a determinant of 3×3 matrix.

$$\det \mathcal{U} = F_1^2(T)$$

This gives

$$F(\zeta) = \left(\sqrt{GUE(\zeta)}\right) \int F_1(T) \left(\sqrt{\det\left(\delta_{ij} - (\alpha_i, \beta_j)\right)_{1 \le i,j \le 3}}\right) dT$$